Long Time Asymptotics for Stochastic PDEs

Wei Liu

Jiangsu Normal University

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Outline

1. Variational Framework

2. Long time asymptotics

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$$du(t) = A(u(t))dt + B(u(t))dW_t$$

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- **Ren/Röckner/Wang [JDE '07]; Zhang [SD '09]**

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  - ${}_V\langle x - y, A(x) - A(y)\rangle_{V^*} \leq \rho(y)\|x - y\|_H^2$

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- **Well-posedness: L./Röckner [JFA '10; JDE '13 $\cdots$]**
  - Burgers equation/2D Navier-Stokes equation
  - power law fluids
  - MHD equation
  - 2D Boussinesq model
  - 3D Leray-α model
  - 2D magnetic Bénard problem
  - Cahn-Hilliard equation
  - Kuramoto-Sivashinsky/Swift-Hohenberg equations $\cdots$

Well-posedness for various models

- **Es-Sarhir/Renesse [SIAM '11]**: mean curvature flow
- **Röckner/Zhu/Zhu [NA '15]** : functional SPDE
- **Neelima/Siska [Stochastics '20]** : improved coercivity
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- **Brzezniak/L./Zhu [NA '14]**: Lévy noise
- **L./Röckner/Silva [SIAM '18; JFA '21]** : time-fractional SPDE
- **Hong/Hu/L. [AAP '23]**: Mckean-Vlasov SPDE $\cdots$

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Long time behavior

Two paths of investigation:

(i) **Distribution: ergodicity** $\cdots$

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existence, dimension, structure $\cdots$

Ergodicity // Harnack inequality

- **Barbu/Da Prato [AMO '06]**: SPME
- **Wang [AOP '07]**: Harnack inequality for SPME
- **L. [JEE '09]**: monotone SPDE
- **Xu [JEE '11]** : SNSE/asymptotic log-Harnack inequality
- **Wang [DCDS '15]** : exponential convergence rate/monotone
- **Dareiotis/Gess/Tsatsoulis [SIAM '20]** : multiplicative noise
- **Hong/Li/L. [JEE, PotA '21]** : L. M./degenerate noise
- **Huang et al [JDE '22]**: exponential mixing $\cdots$

Random attractor

- **Beyn/Gess/Lescot/Röckner [CPDE '11]**: SPME
- **Gess/L./Röckner [JDE '11]**: monotone SPDE
- **Gess [JDE, JDDE '13]** : degenerate/singular SPDE
- **Gess [AOP '14]** : SPME/linear multiplicative noise
- **Flandoli/Gess/Scheutzow [AOP '17]** : SPME/synchronization
- **Gess/L./Schenke [JDE '20]** : locally monotone SPDE
- **Guo/L./Nguyen/Wang [Math. Ann. '23]**: dimension estimate

Long time behavior via random attractor

Intuitively (A monotone/locally monotone)

$$S(t,s;\omega)x = x + \int_s^t A(S(r,s;\omega)x)\mathrm{d}r + N_t(\omega) - N_s(\omega).$$

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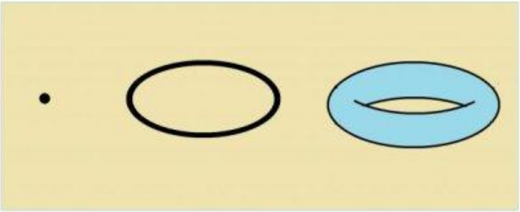
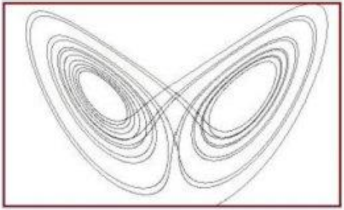
(3) $\mathcal{D}$ is attractive: $d_H(S(t,\cdot;\omega)I,\ \mathcal{D}(\theta_t\omega)) \to 0$ as $t \to \infty$.

Examples of Attractor

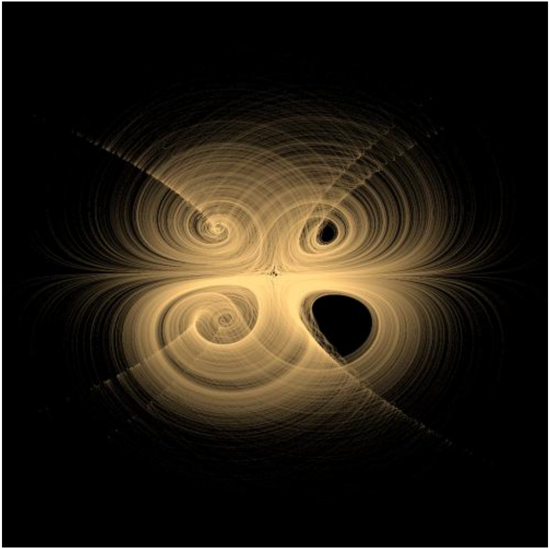


三种经典吸引子



奇异吸引子

Attractors



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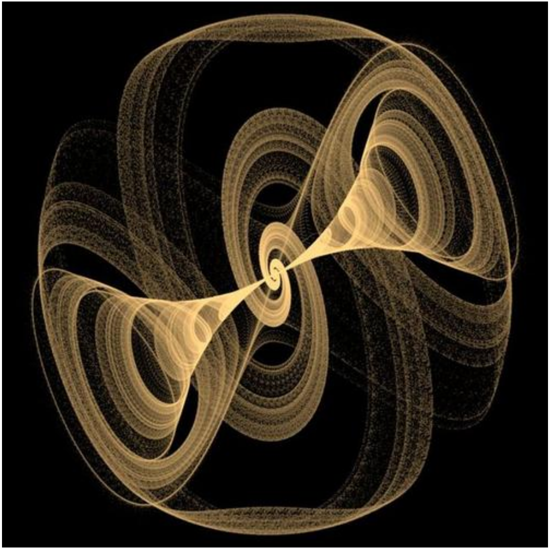

Concrete example

Consider the stochastic p-Laplace equation $(p > 2, \lambda > 0)$

$$dX_t = \left[\mathbf{div}\left(|\nabla X_t|^{p-2}\nabla X_t\right) + \lambda X_t\right] dt + \text{additive noise} \tag{1}$$

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(iv) In deterministic case, the attractor of (1) is ∞ dimension.

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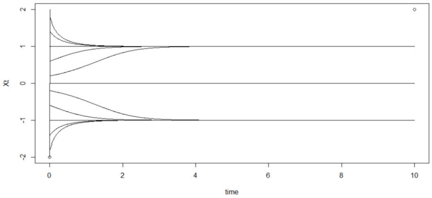
Remark

Those three equations are typical SPDE examples in variational framework.

Example (simulation by Gess)

$$dX_t = (X_t - X_t^3)dt \quad \Longrightarrow \quad dX_t = (X_t - X_t^3)dt + \sigma dW_t$$



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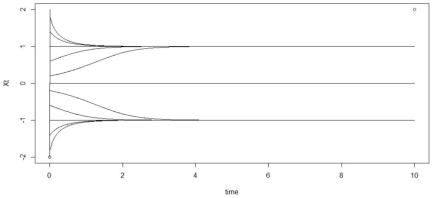

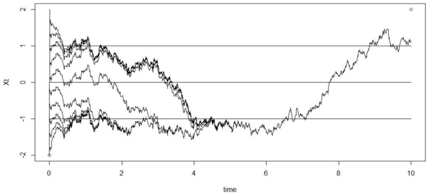

Random attractors

For random attractor, we consider the following additive noise case

$$dX_t = A(X_t)dt + dN_t,$$

where $A : V \to V^*$ and N_t is a Lévy type noise in H satisfying some moment condition.

Recall the **Gelfand triple**

$$V \subset H(\equiv H^*) \subset V^*.$$

Main assumptions: local monotone

Suppose that for some $\alpha \geq 2$ and $\beta \geq 0$ with $\beta(\alpha - 1) \leq 2$, we have

$(H1)$ The map $s \mapsto {}_{V^*}\langle A(v_1 + sv_2), v\rangle_V$ is continuous on $\mathbb{R}$.

$(H2)$ $2_{V^*}\langle A(v_1) - A(v_2), v_1 - v_2\rangle_V \leq (C + \eta(v_1) + \rho(v_2)) \|v_1 - v_2\|_H^2$,

where $\rho, \eta : V \to [0, +\infty)$ are locally bounded measurable functions.

$(H3)$ $2_{V^*}\langle A(v), v\rangle_V \leq -\gamma \|v\|_V^\alpha + K\|v\|_H^2 + C$.

$(H4)$ $\|A(v)\|_{V^*}^{\frac{\alpha}{\alpha-1}} \leq C\left(1 + \|v\|_V^\alpha\right)\left(1 + \|v\|_H^\beta\right)$.

Existence of random attractor

Theorem [Gess/L./Schenke, JDE '20]

Suppose that $V \subseteq H$ is compact and A satisfies $(H1)$-$(H4)$. Let $S(t,s;\omega)$ be the (associated) continuous cocycle. Then

(i) $S(t,s;\omega)$ is a compact cocycle.

For $\alpha = 2$ we assume $K < \frac{\gamma\lambda}{4}$ in $(H3)$. Then

(ii) there is a random attractor $\mathcal{D}$ for $S(t,s;\omega)$.

Application to Examples

Some typical examples:

$$A(u) = \Delta(|u|^{r-1}u), \ r > 1 \ \ \text{(porous media equation);}$$

$$A(u) = \textbf{div}(|\nabla u|^{p-2}\nabla u), \ p > 1 \ \ \text{(p-Laplace equation);}$$

$$A(u) = \frac{\partial^2 u}{\partial x^2} - f(u)\frac{\partial u}{\partial x}, \quad \text{(Burgers type equation);}$$

$$A(u) = \Delta u - (u \cdot \nabla)u, \quad \text{(2D Navier-Stokes equation)}$$

$$A(u) = \text{div}\,(\tau(u)) - (u \cdot \nabla)u \ \ \text{(non-Newtonian fluids)}$$

Further Examples

$\diamondsuit$ Stochastic 3D Leray-α model;

$\diamondsuit$ Stochastic MHD (magneto-Hydrodynamic) equation;

$\diamondsuit$ Stochastic 2D Boussinesq model;

$\diamondsuit$ Stochastic 2D magnetic Bénard problem;

$\diamondsuit$ Stochastic Ladyzhenskaya model

$\diamondsuit$ Stochastic tamed 3D Navier-Stokes equation

$\diamondsuit$ Stochastic Kuramoto-Sivashinsky type equations

$\diamondsuit$ Stochastic Cahn-Hilliard type equations $\cdots$

Ergodicity: local monotone

$$dX_t = (LX_t + B(X_t, X_t) + R(X_t))dt + GdW_t,$$

where L (e.g. Δ) is a self-adjoint linear operator on H, $B(\cdot,\cdot): V \times V \to V^*$ ($V := \mathrm{Dom}(L^{1/2})$) is a bilinear continuous map, $R: H \to H$ is a bounded linear operator.

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Theorem [Hong/Li/L., J. Evol. Equ. '21]

Under certain assumptions on G (highly degenerate noise),

$$\mathrm{Ran}(G) = P_{N_0}H, \quad Gu = 0 \text{ for } u \in (I - P_{N_0})H,$$

Then asymptotic log-Harnack inequality holds ($\Longrightarrow$ ergodicity).

Finite Dimension of the attractor

$$A(u) = Lu + B(u,u) + R(t,u) + f(t), \quad GdW_t = hd\beta_t$$

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Theorem [Guo/L./Nguyen/Wang, Math. Ann. '23]

Under certain assumptions, the fractal dimension of (random) attractor $\mathcal{D}$ has a finite upper bound in H (also in V)

$$\dim_H \mathcal{D} \le \frac{3m}{\ln 2} \ln\left(8\sqrt{m} + 1\right) < \infty.$$

Key point

(H) Lipschitz projection in H: there exist $t_0, \lambda > 0$, random variables $C_i(\omega) \geq 0$ and m-dimensional projector $P_m : H \to P_m H$ such that for any $x, y \in \mathcal{D}$,

$$\|P_m u(t_0,\tau,\omega)x - P_m u(t_0,\tau,\omega)y\|_H \leq e^{\int_0^{t_0} C_0(\theta_s\omega)ds}\|x-y\|_H,$$

$$\|(I-P_m)u(t_0,\tau,\omega)x - (I-P_m)u(t_0,\tau,\omega)y\|_H$$

$$\leq \left(e^{-\lambda t_0 + \int_0^{t_0} C_1(\theta_s\omega)ds} + \frac{1}{8}e^{\int_0^{t_0} C_0(\theta_s\omega)ds}\right)\|x-y\|_H.$$

♣ Foias, Temam, Vishik; Debussche, Crauel/Flandoli, Wang, Zhou $\cdots$

Acknowledgement

Thank you for your attention!